Good Grosshans filtration in a family

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Abstract

We reprove the main result of our joint work [17], with the base field replaced by a commutative noetherian ring $\mathbf{k}$. This has repercussions for the cohomology $H^*(G, A)$ of a reductive group scheme G over $\mathbf{k}$, with coefficients in a finitely generated commutative $\mathbf{k}$-algebra A. For clarity we shamelessly copy from [17].

1 Introduction

Let $\mathbf{k}$ be a noetherian ring. Consider a flat linear algebraic group scheme G defined over $\mathbf{k}$. Recall that G has the cohomological finite generation property (CFG) if the following holds: Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally by $\mathbf{k}$-algebra automorphisms. (So G acts from the right on $\mathrm{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a $\mathbf{k}$-algebra. Here, as in [11, I.4], we use the cohomology introduced by Hochschild, also known as 'rational cohomology'.

This note is part of the project of studying (CFG) for reductive G. More specifically, the intent of this note is to generalize the main result of [17] to the case where the base ring of GL_N is our noetherian ring $\mathbf{k}$.That will allow to enlarge the scope of several results in [18], [7]. Let us give an example. Let G be a reductive group scheme over $\mathrm{Spec}(\mathbf{k})$ in the sense of SGA3. Recall this means that G is affine and smooth over $\mathrm{Spec}(\mathbf{k})$ with geometric fibers that are connected reductive. Let G act rationally by $\mathbf{k}$-algebra automorphisms on a finitely generated commutative $\mathbf{k}$-algebra A. We do not know (CFG) in this generality, but now we can state at least that the $H^m(G, A)$ are noetherian modules over the ring of invariants A^G. And if $\mathbf{k}$ contains a finite ring we do indeed know that $H^*(G, A)$ is a finitely generated $\mathbf{k}$-algebra. See section 10 for these results and related material.

To formulate the main result, let $N \geq 1$ and let G be the affine algebraic group GL_N or SL_N over $\mathbf{k}$. We use notations and terminology as in [17], [7]. Recall in particular that a G-module V module is said to have *good Grosshans filtration* if the embedding $\operatorname{gr} V \to \operatorname{hull}_\nabla(\operatorname{gr} V)$ of Grosshans is an isomorphism [7, Definition 27]. Such a module is G-acyclic. It need not be flat over $\mathbf{k}$. The module V has a good Grosshans filtration if and only if it satisfies the following cohomological criterion: $H^i(G, V \otimes_{\mathbf{k}} \nabla(\lambda))$ vanishes for all $i > 0$ and all dominant weights λ. Over fields this is the familiar criterion for having a good filtration. Indeed over a field there is no difference between 'good filtration' and 'good Grosshans filtration'. But modules with good filtration are required to be free over $\mathbf{k}$ and this is not the right requirement in our present setting. We wish to allow the filtration of V to have an associated graded that is a direct sum of modules of the form $\nabla(\lambda) \otimes_{\mathbf{k}} J(\lambda)$ with G acting trivially on $J(\lambda)$. The $J(\lambda)$ need not be free over $\mathbf{k}$; they even do not have to be flat over $\mathbf{k}$.

Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally by $\mathbf{k}$-algebra automorphisms. Let M be a noetherian A-module on which G acts compatibly. This means that the structure map $A \otimes_{\mathbf{k}} M \to M$ is a G-module map. We also say that M is a (noetherian) AG-module. (Later our convention will be that any AG-module is noetherian.)

Our main theorem is

Theorem 1.1 *If A has a good Grosshans filtration, then there is a finite resolution*

$$0 \to M \to N_0 \to N_1 \to \cdots \to N_d \to 0$$

where the N_i are noetherian AG-modules with good Grosshans filtration.

Corollary 1.2 *The $H^i(G, M)$ are noetherian A^G-modules and they vanish for $i \gg 0$.*

Proof One may compute $H^*(G, M)$ with the resolution $N_0 \to \cdots N_d \to 0$. So the result follows from invariant theory [7, Theorem 12, Theorem 9]. □

Remark 1.3 It is natural to ask if the same results hold for other Dynkin types. For the Corollary the answer is yes, because of Theorem 10.5 below. For Theorem 1.1 we do not know how to keep the N_i noetherian, but otherwise it goes through by [7, Proposition 28] and Theorem 10.5 below.

We will actually prove a more technical version of the theorem. This is the key difference with the proof in [17]. Recall that the fundamental weights $\varpi_1, \ldots, \varpi_N$ are given by $\varpi_i = \sum_{j=1}^{i} \epsilon_j$. Let ρ be their sum and let $\nabla_r = \nabla(r\rho)$.

Proposition 1.4 *If A has a good Grosshans filtration, then*

- $H^i(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ *vanishes for* $i \gg 0$,
- $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ *vanishes for* $r \gg 0$.

Define the 'Grosshans filtration dimension' of M to be the minimum d for which $H^{d+1}(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes. As $(\nabla_r \otimes_{\mathbf{k}} \nabla_r)^G = \mathbf{k}$, we have a natural map $V \to V \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r$ for any G-module V. In the theorem one may start with $N_0 := M \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r$. The cokernel of $M \to N_0$ will then have a lower Grosshans filtration dimension. And Grosshans filtration dimension zero implies good Grosshans filtration [7, Proposition 28].

Remark 1.5 In Proposition 1.4 it would suffice to tensor once with ∇_r. Our formulation is adapted to the proof of Theorem 1.1.

As in [17] the method of proof of Theorem 1.1 is based on the functorial resolution [14] of the ideal of the diagonal in $Z \times Z$ when Z is a Grassmannian of subspaces of $\mathbf{k}^N$. This is used inductively to study equivariant sheaves on a product X of such Grassmannians. That leads to a special case of the theorems, with A equal to the Cox ring of X, multigraded by the Picard group $\mathrm{Pic}(X)$, and M compatibly multigraded. Next one treats cases when on the same A the multigrading is replaced with a 'collapsed' grading with smaller value group and M is only required to be multigraded compatibly with this new grading. Here the trick is that an associated graded of M has a multigrading that is collapsed a little less. The suitably multigraded Cox rings are then used as in [17] to cover the general case 1.1.

Recall that section 10 gives some consequences for earlier work.

2 Recollections and conventions

Some unexplained notations, terminology, properties, ... can be found in [11]. For the time being G is either GL_N or SL_N. Some things are best told with

GL_N, but the conclusion of Proposition 1.4 refers only to the SL_N-module structure.

First let $G = \mathrm{GL}_N$, with B^+ its subgroup of upper triangular matrices, B^- the opposite Borel subgroup, $T = B^+ \cap B^-$ the diagonal subgroup, $U = U^+$ the unipotent radical of B^+. The roots of U are positive, contrary to the Århus convention followed in [7]. The character group $X(T)$ has a basis $\epsilon_1 \ldots, \epsilon_N$ with $\epsilon_i(\mathrm{diag}(t_1, \ldots, t_N)) = t_i$. An element $\lambda = \sum_i \lambda_i \epsilon_i$ of $X(T)$ is often denoted $(\lambda_1, \ldots, \lambda_N)$. It is called a polynomial weight if the λ_i are nonnegative. It is called a dominant weight if $\lambda_1 \geq \cdots \geq \lambda_N$. It is called anti-dominant if $\lambda_1 \leq \cdots \leq \lambda_N$. The fundamental weights $\varpi_1, \ldots, \varpi_N$ are given by $\varpi_i = \sum_{j=1}^{i} \epsilon_j$. If $\lambda \in X(T)$ is dominant, then $\mathrm{ind}_{B^-}^{G}(\lambda)$ is the *dual Weyl module* or *costandard module* $\nabla_G(\lambda)$, or simply $\nabla(\lambda)$, with highest weight λ. The *Grosshans height* of λ is $\mathrm{ht}(\lambda) = \sum_i (N - 2i + 1)\lambda_i$. It extends to a homomorphism $\mathrm{ht} : X(T) \otimes \mathbb{Q} \to \mathbb{Q}$. The determinant representation has weight ϖ_N and one has $\mathrm{ht}(\varpi_N) = 0$. Each positive root β has $\mathrm{ht}(\beta) > 0$. If λ is a dominant polynomial weight, then $\nabla_G(\lambda)$ is called a *Schur module*. If α is a partition with at most N parts then we may view it as a dominant polynomial weight and the *Schur functor* S^α maps $\nabla_G(\varpi_1)$ to $\nabla_G(\alpha)$. (This is the convention followed in [14]. In [1] the same Schur functor is labeled with the conjugate partition $\tilde{\alpha}$.) The formula $\nabla(\lambda) = \mathrm{ind}_{B^-}^{G}(\lambda)$ just means that $\nabla(\lambda)$ is obtained from the Borel-Weil construction: $\nabla(\lambda)$ equals $H^0(G/B^-, \mathcal{L}_\lambda)$ for a certain line bundle $\mathcal{L}_\lambda$ on the flag variety G/B^-.

Now consider the case $G = \mathrm{SL}_N$. There are similar conventions for SL_N-modules. For instance, the costandard modules for SL_N are the restrictions of those for GL_N. The Grosshans height on $X(T)$ induces one on $X(T \cap \mathrm{SL}_N) \otimes \mathbb{Q}$. The *multicone* $\mathbf{k}[\mathrm{SL}_N /U]$ consists of the f in the coordinate ring $\mathbf{k}[\mathrm{SL}_N]$ that satisfy $f(xu) = f(x)$ for $u \in U$. As an SL_N-module it is the direct sum of all costandard modules. It is also a finitely generated algebra [13], [8], [7, Lemma 23]. Note that $\mathbf{k}[\mathrm{SL}_N /U]$ is SL_N-equivariantly isomorphic to $\mathbf{k}[\mathrm{SL}_N /U^-]$, so that here it does not matter whether one follows the Århus convention or not.

Definition 2.1 A *good filtration* of a G-module V is a filtration $0 = V_{\leq -1} \subseteq V_{\leq 0} \subseteq V_{\leq 1} \ldots$ by G-submodules $V_{\leq i}$ with $V = \cup_i V_{\leq i}$, so that its associated graded $\mathrm{gr}\, V$ is a direct sum of costandard modules.

A Schur filtration of a polynomial GL_N-module V is a filtration $0 = V_{\leq -1} \subseteq V_{\leq 0} \subseteq V_{\leq 1} \ldots$ by GL_N-submodules with $V = \cup_i V_{\leq i}$, so that its associated graded $\mathrm{gr}\, V$ is a direct sum of Schur modules. The *Grosshans*

filtration of V is the filtration with $V_{\leq i}$ the largest G-submodule of V whose weights λ all satisfy $\mathrm{ht}(\lambda) \leq i$. Good filtrations and Grosshans filtrations for SL_N-modules are defined similarly. The literature contains more restrictive definitions of good/Schur filtrations. Ours are the right ones when dealing with representations that need not be finitely generated over $\mathbf{k}$.

Let M be a G-module provided with the Grosshans filtration. Recall from [7] that M has *good Grosshans filtration* if the embedding of $\mathrm{gr}\, M$ into $\mathrm{hull}_\nabla(\mathrm{gr}\, M) = \mathrm{ind}_{B^-}^G M^U$ is an isomorphism. Then $\mathrm{gr}\, M$ is a direct sum of modules of the form $\nabla(\lambda) \otimes_{\mathbf{k}} J(\lambda)$ with G acting trivially on $J(\lambda)$. The $J(\lambda)$ need not be flat. If they are all free then we are back at the case of a good filtration.

A G-module M has good Grosshans filtration if and only if $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N / U])$ vanishes [7, Proposition 28]. And $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N / U])$ vanishes if and only if $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} V)$ vanishes for every module V with good filtration. A module with good filtration has good Grosshans filtration and is flat as a $\mathbf{k}$-module. The tensor product of two modules with good filtration has good filtration [11, Lemma B.9, II Proposition 4.21]. The tensor product of a module with good filtration and one with good Grosshans filtration thus has good Grosshans filtration. If $H^i(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N / U])$ vanishes, then so does $H^{i+1}(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N / U])$. This follows from [7, Proposition 28] and dimension shift. Note that $\mathbf{k}$ itself has good filtration. So a module with good Grosshans filtration is SL_N-acyclic, hence also G-acyclic when $G = \mathrm{GL}_N$.

3 Gradings

Let $\Theta = \mathbb{Z}^r$ with standard basis $e_1, \ldots, e_r$. We partially order Θ by declaring that $I \geq J$ if $I_q \geq J_q$ for $1 \leq q \leq r$. The *diagonal* $\mathrm{diag}(\Theta)$ consists of the integer multiples of the vector $E = (1, \ldots, 1)$. By a *good G-algebra* we mean a finitely generated commutative $\mathbf{k}$-algebra A on which G acts rationally by $\mathbf{k}$-algebra automorphisms so that A has a good filtration as a G-module. We say that A is a *good $G\Theta$-algebra* if moreover A is Θ-graded by G-submodules,

$$A = \bigoplus_{I \in \Theta,\ I \geq 0} A_I$$

with

- $A_I A_J \subset A_{I+J}$,

- A is generated over A_0 by the A_{e_q},
- G acts trivially on A_0.

Motivated by the Segre embedding we define

$$\operatorname{diag}(A) = \bigoplus_{I \in \operatorname{diag}(\Theta)} A_I$$

and $\operatorname{Proj}(A) := \operatorname{Proj}(\operatorname{diag}(A))$. By an AG-module we will mean a noetherian A-module M with compatible G-action. If moreover M is Θ-graded by G-submodules M_I so that $A_I M_J \subset M_{I+J}$, then we call M an *$AG\Theta$-module*.

Definition 3.1 We call a G-module M *negligible* if

- $H^i(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes for $i \gg 0$ and
- $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes for $r \gg 0$.

In other words, M must have finite Grosshans filtration dimension and $M \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r$ must have good Grosshans filtration for $r \gg 0$. We will be interested in AG-modules being negligible. In particular a good $G\Theta$-algebra A is itself negligible.

Lemma 3.2 *Let*

$$0 \to M' \to M \to M'' \to 0$$

be exact, with M' be negligible. Then M is negligible iff M'' is negligible.

Proof Use that if $H^i(\mathrm{SL}_N, V \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes, then so does $H^{i+1}(\mathrm{SL}_N, V \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$. $\square$

Lemma 3.3 *Let $0 \to M_0 \to M_1 \to \cdots \to M_q \to 0$ be a complex of AG-modules whose homology modules $\ker(M_i \to M_{i+1})/\operatorname{im}(M_{i-1} \to M_i)$ are negligible for $i = 0, \ldots, q$. If M_i is negligible for $i < q$ then M_q is negligible.* $\square$

4 Picard graded Cox rings

If V is a finitely generated projective $\mathbf{k}$-module, we denote its dual by $V^{\#}$. For $1 \leq s \leq N$, let $\mathrm{Gr}(s)$ be the Grassmannian scheme over $\mathbf{k}$ parametrizing rank s subspaces of the dual $\nabla(\varpi_1)^{\#}$ of the defining representation of GL_N. If one does base change to an algebraically closed field $\mathbb{F}$, then one gets the Grassmannian variety $\mathrm{Gr}(s)_{\mathbb{F}}$ over $\mathbb{F}$ parametrizing s-dimensional subspaces of the dual $\nabla(\varpi_1)^{\#}$ of the defining representation of GL_N. We think of $\mathrm{Gr}(s)$ as a constant family parametrized by $\mathrm{Spec}(\mathbf{k})$. Note that we often suppress the base ring $\mathbf{k}$ in the notation. The point is that we will argue in a manner which minimizes the need to pay attention to the base ring. Let $\mathcal{O}(1)$ denote the usual ample sheaf on $\mathrm{Gr}(s)$, corresponding with a generator of the Picard group of $\mathrm{Gr}(s)_{\mathbb{F}}$. We wish to view it as a G-equivariant sheaf. To this end consider $G = \mathrm{GL}_N$ with its parabolic subgroup $P = \{\, g \in G \mid g_{ij} = 0 \text{ for } i > N-s,\ j \leq N-s \,\}$ and identify $\mathrm{Gr}(s)$ with G/P. Then a G-equivariant vector bundle is the associated bundle of its fiber over P/P, where this fiber is a P-module. For the line bundle $\mathcal{O}(1)$ we let P act by the weight $\varpi_N - \varpi_{N-s}$ on the fiber over P/P. With this convention $\Gamma(\mathrm{Gr}(s), \mathcal{O}(1))$ is the Schur module $\nabla(\varpi_s)$, cf. [11, II.2.16]. More generally, for $n \geq 0$ one has $\Gamma(\mathrm{Gr}(s), \mathcal{O}(n)) = \nabla(n\varpi_s)$. So

$$A\langle s\rangle = \oplus_{n\geq 0}\Gamma(\mathrm{Gr}(s), \mathcal{O}(n))$$

is a good $G\mathbb{Z}$-algebra. Recall that $\Theta = \mathbb{Z}^r$. Let $1 \leq s_i \leq N$ be given for $1 \leq i \leq r$. Repetitions are definitely allowed. Then the Cox ring $A\langle s_1\rangle \otimes \cdots \otimes A\langle s_r\rangle$ of $\mathrm{Gr}(s_1) \times \cdots \mathrm{Gr}(s_r)$ is a good $G\Theta$-algebra. We put $C = C_0 \otimes A\langle s_1\rangle \otimes \cdots \otimes A\langle s_r\rangle$, where C_0 is a polynomial algebra on finitely many generators over $\mathbf{k}$ with trivial G-action. Then C is also a good $G\Theta$-algebra. Here G may be either SL_N or GL_N. We wish to prove

Proposition 4.1 *Every $CG\Theta$-module is negligible.*

The proof will be by induction on the rank r of Θ. It will be finished in 6.6. Notice that the property of being negligible depends only on the SL_N-module structure. In particular, a shift in the grading makes no difference. As base of the induction we use

Lemma 4.2 *A CG-module M that is noetherian over C_0 is negligible.*

Proof We may view M as C_0G-module and forget that M is a C-module. Say $G = \mathrm{SL}_N$. First let us show that $H^i(G, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N / U])$ vanishes for $i \gg 0$. We claim that it only depends on the weights of M how large i must be taken. Say all weights of M have length at most R. We argue by induction on the highest weight of M. To perform the induction, we first choose a total order on weights of length at most R, that refines the usual dominance order of [11, II.1.5]. Initiate the induction with $M = 0$. For the induction step, consider the highest weight μ in M and let M_μ be its weight space. We let G act trivially on M_μ. Now, by [7, Proposition 21] $\Delta(\mu)_{\mathbb{Z}} \otimes_{\mathbb{Z}} M_\mu$ maps to M, and the kernel and the cokernel of this map have lower highest weight. So we still need to see that $\Delta(\mu)_{\mathbb{Z}} \otimes_{\mathbb{Z}} M_\mu$ itself has the required property. But $\nabla(\mu)_{\mathbb{Z}} \otimes_{\mathbb{Z}} M_\mu$ has it by the universal coefficient theorem [3, A.X.4.7], and the natural map from $\Delta(\mu)_{\mathbb{Z}} \otimes_{\mathbb{Z}} M_\mu$ to $\nabla(\mu)_{\mathbb{Z}} \otimes_{\mathbb{Z}} M_\mu$ also has kernel and cokernel of lower highest weight. All in all there is an effective bound for i in terms of the weight structure of M.

Now we still have to show that $H^j(G, M \otimes_{\mathbf{k}} \nabla(r\rho) \otimes_{\mathbf{k}} \nabla(r\rho) \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N / U])$ vanishes for $j > 0$ when r is large enough. First let V be a C_0G-module that is obtained by tensoring a flat noetherian $G_{\mathbb{Z}}$-module $V_{\mathbb{Z}}$ with a C_0-module on which G acts trivially. Then to show that V has the required property we wish to invoke the universal coefficient theorem [3, A.X.4.7] again. We take r so large that $r\rho - \mu$ is dominant for all weights μ of V. Look at the $H^j(G_{\mathbb{Z}}, V_{\mathbb{Z}} \otimes \nabla_{\mathbb{Z}}(r\rho) \otimes \nabla_{\mathbb{Z}}(r\rho) \otimes \nabla_{\mathbb{Z}}(\nu))$ for $j > 0$, ν dominant. They are noetherian $\mathbb{Z}$-modules. The corresponding groups vanish over a field $\mathbb{F}$. Indeed in view of [20] the reasoning in [4, §3] shows that $V_{\mathbb{F}} \otimes \nabla_{\mathbb{F}}(r\rho) \otimes \nabla_{\mathbb{F}}(r\rho)$ has good filtration because r is so large that $r\rho - \mu$ is dominant for all weights μ of V. But then the above $H^j(\ldots)$ must vanish over $\mathbb{Z}$ for such r. So $H^j(G, V \otimes \nabla(r\rho) \otimes \nabla(r\rho) \otimes \mathbf{k}[\mathrm{SL}_N / U])$ vanishes for $j > 0$ by the universal coefficient theorem. Now one may argue by induction on the highest weight of M again. $\square$

Remark 4.3 This kind of reasoning with the universal coefficient theorem is needed in many places to extend facts from fields to our base ring $\mathbf{k}$. We may use it tacitly.

Notation 4.4 For $1 \leq q \leq r$ we denote by $C^{\widehat{q}}$ the subring $\bigoplus_{I_q = 0} C_I$.

We further assume $r \geq 1$. The inductive hypothesis then gives:

Lemma 4.5 *Let $1 \leq q \leq r$. If the $CG\Theta$-module M is noetherian over the subring $C^{\widehat{q}}$, then M is negligible.*

5 Coherent sheaves

We now have $\mathrm{Proj}(C) = \mathrm{Spec}(C_0) \times \mathrm{Gr}(s_1) \times \cdots \mathrm{Gr}(s_r)$. Call the projections of $\mathrm{Proj}(C)$ onto its respective factors $\pi_0, \ldots, \pi_r$. For $I \in \Theta$ define the coherent sheaf $\mathcal{O}(I) = \bigotimes_{i=1}^r \pi_i^*(\mathcal{O}(I_i))$. So $C = \bigoplus_{I \geq 0} \Gamma(\mathrm{Proj}(C), \mathcal{O}(I))$. For a $CG\Theta$-module M let $M^\sim$ be the coherent G-equivariant [11, II F.5] sheaf on $\mathrm{Proj}(C)$ constructed as in [10, II 5.11] from the $\mathbb{Z}$-graded module $\mathrm{diag}(M) := \bigoplus_{I \in \mathrm{diag}(\Theta)} M_I$.

Conversely, to a coherent sheaf $\mathcal{M}$ on $\mathrm{Proj}(C)$, we associate the Θ-graded C-module

$$\Gamma_*(\mathcal{M}) = \bigoplus_{I \geq 0} \Gamma(\mathrm{Proj}(C), \mathcal{M}(I)),$$

where $\mathcal{M}(I) = M \otimes \mathcal{O}(I)$. We also put $H_*^t(\mathcal{M}) = \bigoplus_{I \geq 0} H^t(\mathrm{Proj}(C), \mathcal{M}(I))$.

Lemma 5.1 (Künneth) *Let X and Y be flat projective schemes over an affine scheme $S = \mathrm{Spec}(R)$. Let $\mathcal{F}$ be a quasicoherent sheaf on X and $\mathcal{G}$ one on Y. Assume $\mathcal{F}$, $\mathcal{G}$ are flat over S and that $\Gamma(X, \mathcal{F})$ is flat over R. If $H^i(X, \mathcal{F})$ vanishes for $i > 0$, then $H^i(X \otimes_S Y, \mathcal{F} \boxtimes \mathcal{G}) = \Gamma(X, \mathcal{F}) \otimes_R H^i(Y, \mathcal{G})$ for all i.*

Proof Use [12, Theorem 14]. □

Lemma 5.2 *If $\mathcal{M}$ is a G-equivariant coherent sheaf on $\mathrm{Proj}(C)$, then the $H_*^t(\mathcal{M})$ are $CG\Theta$-modules.*

Proof So we have to show that $H_*^t(\mathcal{M})$ is noetherian as a C-module. Observe that $\mathrm{Proj}(C)$ has a finite affine cover, so that $H_*^t(\mathcal{M})$ vanishes for t large. So we argue by descending induction on t. Assume the result for all larger values of t. By Kempf vanishing $\bigoplus_{q \geq 0} \bigoplus_{n \geq 0} H^q(Gr(s), \mathcal{O}(i+n))$ is a noetherian $\bigoplus_{n \geq 0} \Gamma(Gr(s), \mathcal{O}(n))$ module, for any $i \in \mathbb{Z}$. Similarly, by Lemma 5.1 we see that $\bigoplus_{q \geq 0} H_*^q(\mathrm{Proj}(C), \mathcal{O}(I))$ is a noetherian C-module for any $I \in \Theta$, generated by the $H^q(\mathrm{Proj}(C), \mathcal{O}(I+J))$ with $0 \leq J_j \leq |I_j|$. Now write $\mathcal{M}$ as a quotient of some $\mathcal{O}(iE)^a$ and use the long exact sequence

$$\cdots \to H_*^t(\mathcal{O}(iE)^a) \to H_*^t(\mathcal{M}) \to H_*^{t+1}(\ldots) \to \cdots$$

to finish the induction step. $\square$

Notation 5.3 If M is a Θ-graded module and $I \in \Theta$, then $M(I)$ is the Θ-graded module with $M(I)_J = M_{I+J}$. Further $M_{\geq I}$ denotes $\bigoplus_{J \geq I} M_J$.

Lemma 5.4 *If $I \geq 0$, then the ideal $C_{\geq I}$ of C is generated by C_I.*
If M is a $CG\Theta$-module with $M_{nE} = 0$ for $n \gg 0$, then $M_{\geq nE} = 0$ for $n \gg 0$.

Proof The ideal is generated by C_I because C is generated over C_0 by the C_{e_i}. Let $m \in M_I$. Choose $J \geq 0$ with $I + J \in \mathrm{diag}(\Theta)$. Then mC_{J+qE} vanishes for $q \gg 0$, so $(mC)_{\geq I+J+qE} = 0$ for $q \gg 0$. Now use that M is finitely generated over C. $\square$

Lemma 5.5 *If M is a $CG\Theta$-module, then there is an n_0 so that if $I = nE = (n, \ldots, n) \in \Theta$ with $n > n_0$, then $M_{\geq I} = \Gamma_*(M^{\sim})_{\geq I}$.*

Proof Recall [10, II Exercise 5.9] that we have a natural map $\mathrm{diag}(M) \to \mathrm{diag}(\Gamma_*(M^{\sim}))$ whose kernel and cokernel live in finitely many degrees. Consider the maps $f : \mathrm{diag}(M) \otimes_{\mathrm{diag}(C)} C \to M$ and $g : \mathrm{diag}(M) \otimes_{\mathrm{diag}(C)} C \to \Gamma_*(M^{\sim})$. If N is the kernel or cokernel of f or g then $N_{nE} = 0$ for $n \gg 0$. Now apply the previous lemma. $\square$

Lemma 5.6 *If M is a $CG\Theta$-module and $I \in \Theta$, then $M/M_{\geq I}$ is negligible.*

Proof As M is finitely generated over C, there is $J < I$ with $M = M_{\geq J}$. Now note that for $1 \leq q \leq r$ and $K \in \Theta$ the module $M_{\geq K}/M_{\geq K+e_q}$ is negligible by 4.5. $\square$

Lemma 5.7 *If M is a $CG\Theta$-module and $I \in \Theta$, then M is negligible iff $M_{\geq I}$ is negligible.*

Proof As a $\mathbf{k}G$-module M is the direct sum of $M/M_{\geq I}$ and $M_{\geq I}$. $\square$

Definition 5.8 In view of the above we call an equivariant coherent sheaf $\mathcal{M}$ on $\mathrm{Proj}(C)$ negligible when $\Gamma_*(\mathcal{M})$ is negligible.

The following Lemma is now clear:

Lemma 5.9 *Let $I \in \Theta$. A G-equivariant coherent sheaf $\mathcal{M}$ on* $\mathrm{Proj}(C)$ *is negligible if and only if $\mathcal{M}(I)$ is negligible.*

Lemma 5.10 *Let*

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

be an exact sequence of G-equivariant coherent sheaves on $\mathrm{Proj}(C)$. *There is $I \in \Theta$ with*

$$0 \to \Gamma_*(\mathcal{M}')_{\geq I} \to \Gamma_*(\mathcal{M})_{\geq I} \to \Gamma_*(\mathcal{M}'')_{\geq I} \to 0$$

exact.

Proof The line bundle $\mathcal{O}(E)$ is ample. Apply Lemma 5.4 to the homology sheaves of the complex

$$0 \to \Gamma_*(\mathcal{M}') \to \Gamma_*(\mathcal{M}) \to \Gamma_*(\mathcal{M}'') \to 0.$$

□

Lemma 5.11 *For every $I \in \Theta$ the sheaf $\mathcal{O}(I)$ is negligible.*

Proof Use that C is negligible. □

6 Resolution of the diagonal

We write $X = \mathrm{Proj}(C)$, $Y = \mathrm{Proj}(C^{\hat{r}})$, $Z = \mathrm{Gr}(s)$, where $s = s_r$. So $X = Y \times Z$. We now recall the salient facts from [14], [16] about the functorial resolution of the diagonal in $Z \times Z$. The fact that our base ring is now $\mathbf{k}$ is not a problem. In [14] one already works over a noetherian base, and [16] is just extra. But let us temporarily take Z to be the Grassmannian $\mathrm{Gr}(s)_{\mathbb{Z}}$ over $\mathbb{Z}$. Later we will do the base change from $\mathbb{Z}$ to $\mathbf{k}$. As Z is the Grassmannian that parametrizes the s-dimensional subspaces of $\nabla(\varpi_1)^{\#}$, we have the tautological exact sequence of G-equivariant vector bundles on Z:

$$0 \to \mathcal{S} \to \nabla(\varpi_1)^{\#} \otimes \mathcal{O}_Z \to \mathcal{Q} \to 0,$$

where $\mathcal{S}$ has as fiber above a point the subspace V that the point parametrizes, and $\mathcal{Q}$ has as fiber above this same point the quotient

$\nabla(\varpi_1)^\#/V$. Let π_1, π_2 be the respective projections $Z \times Z \to Z$. Then the composite of the natural maps $\pi_1^*(\mathcal{S}) \to \nabla(\varpi_1)^\# \otimes \mathcal{O}_{Z\times Z}$ and $\nabla(\varpi_1)^\# \otimes \mathcal{O}_{Z\times Z} \to \pi_2^*(\mathcal{Q})$ defines a section of the vector bundle $\mathcal{H}om(\pi_1^*(\mathcal{S}), \pi_2^*(\mathcal{Q}))$ whose zero scheme is the diagonal $\mathrm{diag}(Z)$ in $Z \times Z$. Dually, we get an exact sequence $\mathcal{H}om(\pi_2^*(\mathcal{Q}), \pi_1^*(\mathcal{S})) \to \mathcal{O}_{Z\times Z} \to \mathcal{O}_{\mathrm{diag}\, Z} \to 0$, where $\mathcal{O}_{\mathrm{diag}\, Z}$ is the quotient by the ideal sheaf defining the diagonal. As the rank d of the vector bundle $\mathcal{E} = \mathcal{H}om(\pi_2^*(\mathcal{Q}), \pi_1^*(\mathcal{S}))$ equals the codimension of $\mathrm{diag}(Z)$ in $Z \times Z$, the Koszul complex

$$0 \to \bigwedge^d \mathcal{E} \to \cdots \to \mathcal{E} \to \mathcal{O}_{Z\times Z} \to \mathcal{O}_{\mathrm{diag}\, Z} \to 0$$

is exact. Now each $\bigwedge^i \mathcal{E}$ has a finite filtration whose associated graded is

$$\bigoplus S^\alpha \pi_1^*(\mathcal{S}) \otimes (S^{\tilde\alpha} \pi_2^*(\mathcal{Q}))^\#,$$

where α runs over partitions of i with at most $\mathrm{rank}(\mathcal{S})$ parts, so that moreover the conjugate partition $\tilde\alpha$ has at most $\mathrm{rank}(\mathcal{Q})$ parts. Now do the base change from $\mathbb{Z}$ to $\mathbf{k}$, so that Z is defined over $\mathbf{k}$. The Koszul complex remains exact as it was flat over $\mathbb{Z}$. The expression for the associated graded of $\bigwedge^i \mathcal{E}$ also remains valid.

Plan Now the plan is this: Let $\pi_{1,2}$ be the projection of $Y \times Z \times Z$ onto the product $Y \times Z$ of the first two factors, let π_2 be the projection onto the middle factor Z, and so on. If M is a $CG\Theta$-module, tensor the pull-back along $\pi_{2,3}$ of the Koszul complex with $\pi_{1,3}^*(M^\sim)$, take a high Serre twist and then the direct image along $\pi_{1,2}$ to X. On the one hand $(\pi_{1,2})_*(\pi_{1,3}^*(M^\sim) \otimes \mathcal{O}_{\mathrm{diag}\, Z})$ is just $M^\sim$, but on the other hand the salient facts above allow us to express it in terms of negligible $CG\Theta$-modules. This will prove that M is negligible. We now proceed with the details.

Remark 6.1 Instead of functorially resolving the diagonal in $Z \times Z$, we could have functorially resolved the diagonal in $X \times X$.

Notation 6.2 On a product like $Y \times Z$ an exterior tensor product $\pi_1^*(\mathcal{F}) \otimes \pi_2^*(\mathcal{M})$ is denoted $\mathcal{F} \boxtimes \mathcal{M}$.

Lemma 6.3 *Let $\mathcal{F}$ be a G-equivariant coherent sheaf on Y, and α a partition of i with at most s parts, $i \geq 0$. The sheaf $\mathcal{F} \boxtimes S^\alpha(\mathcal{S})$ on $X = Y \times Z$ is negligible.*

Proof By the inductive assumption

$$\Gamma_*(\mathcal{F}) = \bigoplus_{I\in\mathbb{Z}^{r-1},\ I\geq 0} \Gamma(Y, \mathcal{F}(I))$$

is a $C^{\widehat{r}}$-module with finite Grosshans filtration dimension and $\nabla_r\otimes\nabla_r\otimes\Gamma_*(\mathcal{F})$ has good Grosshans filtration for $r \gg 0$. The vector bundle $\mathcal{S}$ on $Z = G/P$ is associated with the irreducible P-representation with lowest weight $-\epsilon_{N-s+1}$. This representation may be viewed as $\mathrm{ind}_{B^+}^P(-\epsilon_{N-s+1})$, where $-\epsilon_{N-s+1}$ also stands for the one dimensional B^+ representation with weight $-\epsilon_{N-s+1}$. Say $\rho : P \to P^-$ is the isomorphism which sends a matrix to its transpose inverse. Then $\mathrm{ind}_{B^+}^P(-\epsilon_{N-s+1}) = \rho^* \mathrm{ind}_{B^-}^{P^-}(\epsilon_{N-s+1})$. One finds that $S^\alpha(\mathcal{S})$ is associated with $\rho^* \mathrm{ind}_{B^-}^{P^-}(\sum_i \alpha_i\epsilon_{N-s+i}) = \mathrm{ind}_{B^+}^P(-\sum_i \alpha_i\epsilon_{N-s+i})$. (This is the rule $S^\alpha(\nabla_{\mathrm{GL}_s}(\varpi_1)) = \nabla_{\mathrm{GL}_s}(\alpha)$ in disguise.) Then $S^\alpha(\mathcal{S})(n)$ is associated with $\mathrm{ind}_{B^+}^P(-\sum_i \alpha_i\epsilon_{N-s+i} + n\varpi_N - n\varpi_{N-s})$. For $n \geq \alpha_1$ the weight $-\sum_i \alpha_i\epsilon_{N-s+i} + n\varpi_N - n\varpi_{N-s}$ is an anti-dominant polynomial weight, so $\sum_{n\geq\alpha_1} \Gamma(Z, S^\alpha(\mathcal{S})(n))$ has a good filtration by transitivity of induction [11, I.3.5, I.5.12]. Moreover, we have $H^i(Z, S^\alpha(\mathcal{S})(n)) = 0$ for $n \geq \alpha_1$, $i > 0$, by the universal coefficient theorem [3, A.X.4.7], Kempf vanishing [11, Proposition II.4.6 (c)] and the spectral sequence for $\mathrm{ind}_P^G \circ \mathrm{ind}_{B^+}^P$ [11, Proposition I.4.5(c)]. Fix $I = (0, \ldots, 0, \alpha_1)$ and consider $\Gamma_*(\mathcal{F} \boxtimes S^\alpha(\mathcal{S}))_{\geq I}$. By [12, Theorem 12] we may rewrite it as $\sum_{n\geq\alpha_1} \Gamma(Z, \Gamma_*(\mathcal{F}) \otimes_{\mathbf{k}} S^\alpha(\mathcal{S})(n))$, which equals $\Gamma_*(\mathcal{F}) \otimes_{\mathbf{k}} \sum_{n\geq\alpha_1} \Gamma(Z, S^\alpha(\mathcal{S})(n))$ by the universal coefficient theorem. So $\Gamma_*(\mathcal{F} \boxtimes S^\alpha(\mathcal{S}))_{\geq I}$ has finite Grosshans filtration dimension and $\nabla_r \otimes \nabla_r \otimes \Gamma_*(\mathcal{F} \boxtimes S^\alpha(\mathcal{S}))_{\geq I}$ has good Grosshans filtration for $r \gg 0$. Apply Lemma 5.7. $\square$

Lemma 6.4 *For $n \gg 0$ the sheaf*

$$(\pi_{12})_* \left(\pi_{13}^*(M^\sim) \otimes \Big(\mathcal{O}(nE) \boxtimes \mathcal{O}(n)\Big) \otimes \pi_{23}^*(\bigwedge^i \mathcal{E}) \right)$$

is negligible.

Proof The sheaf $\mathcal{O}(E) \boxtimes \mathcal{O}(1)$ is ample. So [10, Theorem 8.8] the sheaf in the Lemma has a filtration with layers of the form

$$(\pi_{12})_* \left(\pi_{13}^*(M^\sim) \otimes \Big(\mathcal{O}(nE) \boxtimes \mathcal{O}(n)\Big) \otimes \pi_{23}^*\Big(S^\alpha(\mathcal{S}) \boxtimes \mathcal{G}\Big)\right).$$

Say $f : Y \times Z \to Y$ is the projection. It is flat and by [10, Proposition 9.3] we have $(\pi_{12})_* \circ \pi_{13}^* = f^* \circ f_*$. Now use this and a projection formula for $(\pi_{12})_*$ to rewrite the layer in the form $(\mathcal{F} \boxtimes S^\alpha(\mathcal{S}))(I)$ for some $I \in \Theta$, with I depending on n. $\square$

Lemma 6.5 *The the Koszul complex*

$$0 \to \bigwedge^d \mathcal{E} \to \cdots \to \mathcal{E} \to \mathcal{O}_{Z\times Z} \to \mathcal{O}_{\operatorname{diag} Z} \to 0$$

remains exact when applying $\pi_{13}^*(M^\sim) \otimes \pi_{23}^*(?)$.

Proof One is basically saying that $\pi_{13}^*(M^\sim)$ and $\mathcal{O}_{\operatorname{diag} Z}$ are Tor independent quasi-coherent sheaves on $Z \times Z$. This is local and can be checked by computing in suitable coordinates. We argue more globally. Let j be the isomorphism $Y \times \operatorname{diag} Z \to Y \times Z$ induced by π_{13}. Write $K_\bullet = 0 \to K_d \to \cdots \to K_0 \to 0$ for $0 \to \bigwedge^d \mathcal{E} \to \cdots \to \mathcal{E} \to \mathcal{O}_{Z\times Z} \to 0$. Let $F_\bullet = \cdots \to F_1 \to F_0 \to 0$ be a resolution of $M^\sim$ by vector bundles. Consider the homology of the total complex of the double complex $\pi_{13}^*(F_\bullet) \otimes \pi_{23}^* K_\bullet$. On the one hand this homology is the homology of $\pi_{13}^*(F_\bullet) \otimes \pi_{23}^* \mathcal{O}_{\operatorname{diag} Z}$, which is just $j^*(F_\bullet)$. So it is concentrated in degree zero, with homology $j^*(M^\sim)$ in degree zero. On the other hand it is the homology of $\pi_{13}^*(M^\sim) \otimes \pi_{23}^*(K_\bullet)$. $\square$

End of proof of Proposition 4.1 Proposition 4.1 now follows from

Lemma 6.6 *$M^\sim$ is negligible.*

Proof From the Koszul complex and the two previous Lemmas we conclude [10, Theorem 8.8] that for $n \gg 0$ the sheaf

$$(\pi_{12})_* \Big(\pi_{13}^*(M^\sim) \otimes \Big(\mathcal{O}(nE) \boxtimes \mathcal{O}(n)\Big) \otimes \pi_{23}^*(\mathcal{O}_{\operatorname{diag}(Z)})\Big)$$

is negligible. This sheaf equals $M^\sim(I)$ for some $I \in \Theta$. $\square$

7 Differently graded Cox rings

Let $c : \{1, \ldots, r\} \to \{1, \ldots, q\}$ be surjective. Put $\Lambda = \mathbb{Z}^q$. We have a *contraction map*, also denoted c, from Θ to Λ with $c(I)_j = \sum_{i \in c^{-1}(j)} I_i$.

Through this contraction we can view our Θ-graded C as Λ-graded. We now have the following generalization of Proposition 4.1:

Proposition 7.1 *Every $CG\Lambda$-module is negligible.*

This will be proved by descending induction on q, with fixed r. The case $q = r$ is clear. So let $q < r$ and assume the result for larger values of q. We may assume $c(r-1) = c(r) = q$. (Otherwise rearrange the factors.) Recall $X = \mathrm{Proj}(C)$, $X = Y \times Z$, with $Y = \mathrm{Proj}(C^{\hat{r}})$, $Z = \mathrm{Proj}(A\langle s\rangle)$.

Notation 7.2 Let $\mathfrak{m}$ be the irrelevant ideal $\bigoplus_{i>0} A\langle s\rangle_i$ of $A\langle s\rangle$. If M is a $CG\Lambda$-module, put $M_{\geq i} = \mathfrak{m}^i M$, and $\mathrm{gr}^i M = M_{\geq i}/M_{\geq i+1}$. If $I \in \Lambda$, put $(M_I)_{\geq i} = M_I \cap \mathfrak{m}^i M$, and $\mathrm{gr}^i M_I = (M_I)_{\geq i}/(M_I)_{\geq i+1}$. We put a $\mathbb{Z}^{q+1}$-grading on $\mathrm{gr}\, M = \bigoplus_i \mathrm{gr}^i M$ with

$$(\mathrm{gr}\, M)_I = \mathrm{gr}^{I_{q+1}} M_{(I_1,\ldots,I_{q-1},I_q+I_{q+1})}.$$

In particular all this applies when $M = C$. Then $\mathrm{gr}\, C$ may be identified with C and the $\mathbb{Z}^{q+1}$-grading on $\mathrm{gr}\, C$ is a contracted grading to which the inductive assumption applies. Write $\Phi = \mathbb{Z}^{q+1}$. Then $\mathrm{gr}\, M$ is a $CG\Phi$-module.

Let M be a $CG\Lambda$-module. By the inductive assumption $\mathrm{gr}\, M$ is negligible. So $\mathrm{gr}^i M_I$ is negligible. As the filtration on M_I is finite, it follows that M_I is negligible. But we need to do a little better. We must show that when i and r are so big that $H^i(\mathrm{SL}_N, \mathrm{gr}\, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes and $H^1(\mathrm{SL}_N, \mathrm{gr}\, M \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r \otimes \mathbf{k}[\mathrm{SL}_N/U])$ vanishes, then the same i and r work for all M_I simultaneously. This is clear too, so M is negligible.

This finishes the proof of Proposition 7.1. $\square$

8 Variations on the Grosshans grading

In this section we will be concerned with representations of SL_N. Mutatis mutandis everything also applies to other connected reductive groups. We now write $G = \mathrm{SL}_N$, with subgroups B^+, B^-, T, U defined in the usual manner. (So they are now the intersections with SL_N of the subgroups of GL_N that had these names.) As explained above, the Grosshans graded $\mathrm{gr}\, V$ of an SL_N-module V has a $\mathbb{Z}$-grading. We also need a Λ-graded version, where Λ is the weight lattice of SL_N.

Following Mathieu [15] we choose a second linear height function $E : \Lambda \otimes \mathbb{R} \to \mathbb{R}$ with $E(\alpha) > 0$ for every positive root α, but now with E injective on Λ. We define a total order on weights by first ordering them by Grosshans height, then for fixed Grosshans height by E. With this total order, denoted $\leq$, we put:

Definition 8.1 If V is a G-module, and λ is a weight, then $V_{\leq\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu \leq \lambda$ in the total order. For instance, $V_{\leq 0}$ is the module of invariants V^G. Similarly $V_{<\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu < \lambda$. Note that $V \mapsto V_{\leq\lambda}$ is a truncation functor for a saturated set of dominant weights [11, Appendix A]. So this functor fits in the usual highest weight category picture. We form the Λ-graded module

$$\operatorname{gr}_\Lambda V = \bigoplus_{\lambda\in\Lambda} V_{\leq\lambda}/V_{<\lambda}.$$

Each $\operatorname{gr}_\lambda V = V_{\leq\lambda}/V_{<\lambda}$ has a B^+-socle $(\operatorname{gr}_\lambda V)^U = V_\lambda^U$ of weight λ. We always view V^U as a B^--module through restriction (inflation) along the homomorphism $B^- \to T$. Then $\operatorname{gr}_\lambda V$ embeds naturally in its 'hull' $\operatorname{hull}_\nabla(\operatorname{gr}_\lambda V) = \operatorname{ind}_{B^-}^G V_\lambda^U$. This hull has the same B^+-socle.

If λ is not dominant, then $\operatorname{gr}_\lambda V$ vanishes, because its socle vanishes. Note that $\bigoplus_{\operatorname{ht}(\lambda)=i} \operatorname{gr}_\lambda V$ is the associated graded of a filtration of $\operatorname{gr}_i V$, where $\operatorname{gr}_\lambda V$ refers to a graded component of $\operatorname{gr}_\Lambda V$ and $\operatorname{gr}_i V$ to one of $\operatorname{gr} V$. Both $\operatorname{gr}_\Lambda V$ and $\operatorname{gr} V$ embed into the hull $\operatorname{ind}_{B^-}^G V^U$, which is Λ-graded. But while $\operatorname{gr}_\Lambda V$ is a Λ-graded submodule of the hull, $\operatorname{gr} V$ need only be a $\mathbb{Z}$-graded submodule. Both $\operatorname{gr}_\Lambda V$ and $\operatorname{gr} V$ contain the socle of the hull.

Although $\operatorname{gr}_\Lambda V$ need not coincide with $\operatorname{gr} V$ it shares some properties:

Lemma 8.2 *1. If A is a finitely generated* $\mathbf{k}$*-algebra, so is* $\operatorname{gr}_\Lambda A$.

2. If A has good Grosshans filtration, then $\operatorname{gr}_\Lambda A$ *is isomorphic to* $\operatorname{gr} A$ *as* $\mathbf{k}$*-algebra.*

Proof Let A be a finitely generated $\mathbf{k}$-algebra. By [7, Lemma 25] the subalgebra A^U is a finitely generated $\mathbf{k}$-algebra. But A^U is isomorphic to $(\operatorname{gr}_\Lambda A)^U$, so by [7, Lemma 46] the algebra $\operatorname{gr}_\Lambda A$ is finitely generated. When A has good Grosshans filtration, $\operatorname{gr}_i A$ is already a direct sum of modules of the form $\operatorname{ind}_{B^-}^G V_\lambda$, where B^- acts on V_λ with weight λ. So then passing to

the associated graded of the filtration of $\mathrm{gr}_i A$ makes no difference. And the algebra structure on both $\mathrm{gr}\, A$ and $\mathrm{gr}_\Lambda A$ agrees with the algebra structure on the hull by the next Lemma. □

Lemma 8.3 *Let A have a good Grosshans filtration, so that $\mathrm{gr}\, A = \mathrm{hull}_\nabla(\mathrm{gr}\, A)$. Let R be a $\mathbb{Z}$-graded algebra with G-action. Assume that for each i one has $R_i = (R_i)_{\leq i}$ in the Grosshans filtration. Then every T-equivariant graded algebra homomorphism $R^U \to (\mathrm{gr}\, A)^U$ extends uniquely to a G-equivariant graded algebra homomorphism $R \to \mathrm{gr}\, A$.*

Proof Use that $\mathrm{hull}_\nabla(\mathrm{gr}_\Lambda A)$ is an induced module. □

9 Proof of the main result

Let us now turn to the proof of Theorem 1.1 for SL_N. Return to the notations introduced in section 2. Thus $G = \mathrm{GL}_N$, with T its maximal torus. We assume the SL_N-algebra A has a good Grosshans filtration and M is a noetherian A-module on which SL_N acts compatibly. Put $\Lambda = \mathbb{Z}^{N-1}$ and identify Λ with a sublattice of $X(T)$ by sending $\lambda \in \Lambda$ to $\sum_i \lambda_i \varpi_i$. Also identify Λ with $X(T \cap \mathrm{SL}_N)$ through the restriction $X(T) \to X(T \cap \mathrm{SL}_N)$. Thus a dominant $\lambda \in \Lambda$ gets identified with a polynomial dominant weight. For such λ we may embed $\mathrm{gr}_\lambda A$ or $\mathrm{gr}_\lambda M$ into its hull which is the tensor product of the Schur module $\nabla_G(\lambda)$ with a $\mathbf{k}$-module with trivial G action. On the Schur module $\nabla_G(\lambda)$ the center of G acts through λ. This makes it natural to use the Λ-grading on $\mathrm{gr}_\Lambda A$ and $\mathrm{gr}_\Lambda M$ to extend the action from SL_N to GL_N, making the center of GL_N act through λ on the graded pieces $\mathrm{gr}_\lambda A$ and $\mathrm{gr}_\lambda M$. We do that.

As the algebra $(\mathrm{gr}_\Lambda A)^U = (\mathrm{gr}\, A)^U$ is finitely generated by [7, Lemma 25] it is also generated by finitely many weight vectors. Consider one such weight vector v, say of weight λ. Clearly λ is dominant. If $\lambda = 0$, map a polynomial ring $P_v := \mathbf{k}[x]$ with trivial G-action to $\mathrm{gr}\, A$ by substituting v for x. Also put $D_v := 1$. Next assume $\lambda \neq 0$. Let $\ell = N - 1$ be the rank of Λ. Recall the Cox rings $A\langle i\rangle$ of section 4. Define a T-action on the Λ-graded algebra

$$P = \bigotimes_{i=1}^{\ell} A\langle i\rangle$$

by letting T act on $\bigotimes_{i=1}^{\ell}\Gamma(\mathrm{Gr}(i),\mathcal{O}(m_i))$ through weight $\sum_i m_i\varpi_i$. So now we have a $G\times T$-action on P, and the T-action corresponds with the Λ-grading. Observe that by the tensor product property [11, Ch. G] the algebra P has a good filtration for the G-action. Let D be the scheme theoretic kernel of λ. So D has character group $X(D)=X(T)/\mathbb{Z}\lambda$ and $D=\mathrm{Diag}(X(T)/\mathbb{Z}\lambda)$ in the notations of [11, I.2.5]. The subalgebra $P^{1\times D}$ is a graded algebra with good filtration such that its subalgebra $P^{U\times D}$ contains a polynomial algebra on one generator x of weight $\lambda\times\lambda$. In fact, this polynomial subalgebra contains all the weight vectors in $P^{U\times D}$ whose weight is of the form $\nu\times\nu$. The other weight vectors in $P^{U\times D}$ have weight of the form $\mu\times\nu$ with ν an integer multiple of λ and $\mu<\nu$. These other weight vectors span an ideal in $P^{U\times D}$. By lemma 8.3 one easily constructs a G-equivariant algebra homomorphism $P^{1\times D}\to\mathrm{gr}_\Lambda A$ that maps x to v. Write it as $P_v^{1\times D_v}\to\mathrm{gr}_\Lambda A$, to stress the dependence on v.

The direct product D of the D_v is a diagonalizable group. It acts on the tensor product C of the finitely many P_v. This C is Λ-graded. We have a graded algebra map $C^D\to\mathrm{gr}_\Lambda A$. Observe that $\mathrm{gr}_\lambda A=\nabla(\lambda)\otimes J(\lambda)$ where $J(\lambda)$ is the λ weight space of A^U, but with trivial G-action. The map $\mathrm{gr}_\lambda C^D\to\mathrm{gr}_\lambda A$ is of the form $\nabla(\lambda)\otimes J'(\lambda)\to\nabla(\lambda)\otimes J(\lambda)$ with G acting trivially on $J'(\lambda)$ also. As each $J'(\lambda)\to J(\lambda)$ is surjective, so is $C^D\to\mathrm{gr}_\Lambda A$. We have proved

Lemma 9.1 *There is a graded G-equivariant surjection $C^D\to\mathrm{gr}_\Lambda A$, where the $G\times D$-algebra C is a good $G\Lambda$ algebra as in 7.1.*

Now recall M is a noetherian A-module on which G acts compatibly, meaning that the structure map $A\otimes M\to M$ is a map of G-modules. Form the 'semi-direct product ring' $A\ltimes M$ whose underlying G-module is $A\oplus M$, with product given by $(a_1,m_1)(a_2,m_2)=(a_1a_2,a_1m_2+a_2m_1)$. By 8.2 $\mathrm{gr}_\Lambda(A\ltimes M)$ is a finitely generated algebra, so we get

Lemma 9.2 *$\mathrm{gr}_\Lambda M$ is a noetherian $\mathrm{gr}_\Lambda A$-module.*

This is of course very reminiscent of the proof of the lemma [9, Theorem 16.9] telling that M^G is a noetherian module over the finitely generated k-algebra A^G. We will tacitly use its counterpart for diagonalizable actions, cf. [2], [11, I.2.11].

Now this lemma implies that $C\otimes_{C^D}\mathrm{gr}_\Lambda M$ is a $CG\Lambda$-module, so by Proposition 7.1 we get

Lemma 9.3 *$C \otimes_{C^D} \mathrm{gr}_\Lambda M$ is negligible*

Next we get

Lemma 9.4 *The module $\mathrm{gr}_\Lambda M$ is negligible.*

Proof Extend the D-action on C to $C \otimes_{C^D} \mathrm{gr}_\Lambda M$ by using the trivial action on the second factor. Then we have a $G \times D$-module structure on $C \otimes_{C^D} \mathrm{gr}_\Lambda M$. As D is diagonalizable, C^D is a direct summand of C as a C^D-module [11, I.2.11] and $(C \otimes_{C^D} \mathrm{gr}_\Lambda M)^{1\times D} = \mathrm{gr}_\Lambda M$ is a direct summand of the G-module $C \otimes_{C^D} \mathrm{gr}_\Lambda M$. It follows that $\mathrm{gr}_\Lambda M$ is negligible. □

Proof of Proposition 1.4 Fix i and r so big that $H^i(\mathrm{SL}_N, \mathrm{gr}\, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N /U])$ vanishes and $H^1(\mathrm{SL}_N, \mathrm{gr}\, M \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N /U])$ vanishes. Enumerate the dominant weigths in Λ as λ_0, λ_1, … according to our total order on weights. Note there are only finitely many dominant weights of given Grosshans height in Λ, so that the order type of the set of dominant weights in Λ is indeed just that of $\mathbb{N}$. (This would be false for the set of dominant weights in $X(T)$.) By induction on λ we get vanishing of $H^i(\mathrm{SL}_N, M_{\leq\lambda} \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N /U])$ and $H^1(\mathrm{SL}_N, M_{\leq\lambda} \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N /U])$ with the same the same i and r. As G-cohomology commutes with direct limits, M is negligible. □

Proof of Theorem 1.1 A GL_N-module has good Grosshans filtration if and only its restriction to SL_N has one. One may embed M into $M \otimes_{\mathbf{k}} \nabla_r \otimes_{\mathbf{k}} \nabla_r$ to start the resolution in Theorem 1.1. As the cokernel has a lower Grosshans filtration dimension, the Theorem follows. □

10 Consequences for earlier work

First let $\mathbf{k}$ be a noetherian ring containing a field $\mathbb{F}$ and let $G_\mathbb{F}$ be a geometrically reductive affine algebraic group scheme over $\mathbb{F}$. Write G for the group scheme over $\mathbf{k}$ obtained by base change along $\mathbb{F} \to \mathbf{k}$. Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally by k-algebra automorphisms.

Theorem 10.1 (CFG when the base ring contains a field)
$H^*(G, A)$ is a finitely generated $\mathbf{k}$-algebra.

This is clear from [18] when A is obtained by base change from an $\mathbb{F}$-algebra with rational $G_{\mathbb{F}}$-action. Anyway, let us adapt the proof of [18, Theorem 1.2]. First we will reduce to the case $G = \mathrm{GL}_N$. Embed $G_{\mathbb{F}}$ in some GL_N over $\mathbb{F}$ and observe that the quotient $\mathrm{GL}_N / G_{\mathbb{F}}$ remains affine under base change to $\mathbf{k}$, cf. [11, I.5.5(1), I.5.4(5)]. For group schemes over $\mathbf{k}$ geometric reductivity is no longer the right notion and we use power-reductivity [7] instead.

Lemma 10.2 *Let G be a power-reductive flat affine algebraic group scheme over a ring R. For any commutative R-algebra S the group scheme G_S over S is a power-reductive.*

Proof If M is a module for G_S, then it is also a module for $G = G_R$ and with the same invariants [7, Remark 52]. By [7, Proposition 10] G_R has property (Int), which implies that G_S has property (Int), hence is power reductive. □

Remark 10.3 The first line of the proof of [7, Proposition 10] ignores that [7, Proposition 6] refers to a finitely generated algebra. Indeed this finite generation hypothesis may safely be dropped, as there is no finiteness hypothesis on the module in the definition of power reductivity. But we do not know that any algebra with G action is a union of finitely generated invariant subalgebras. That is because we simply do not know if representations are always locally finite. See [5, Remarque 11.10.1 in the 2011 edition] for things that can go wrong when $\mathbf{k}$ is not noetherian and $\mathbf{k}[G]$ is not a projective $\mathbf{k}$-module.

Proof of Theorem 10.1 So we may argue as in [19, Lemma 3.7] that $H^*(G, A) = H^*(\mathrm{GL}_N, \mathrm{ind}_G^{\mathrm{GL}_N}(A))$, with $\mathrm{ind}_G^{\mathrm{GL}_N}(A) = (A \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{GL}_N])^G$ a finitely generated $\mathbf{k}$-algebra. This shows we may further assume $G = \mathrm{GL}_N$, an algebraic group scheme over $\mathbf{k}$. We may assume the field $\mathbb{F}$ has positive characteristic p. The map $\mathrm{gr}\, A \to \mathrm{hull}_\nabla \mathrm{gr}\, A$ is still p-power surjective by [7, Theorem 29, Proposition 41]. Write $\mathrm{hull}_\nabla \mathrm{gr}\, A$ as a quotient of an algebra $\mathbf{k} \otimes_{\mathbb{F}_p} R$, where R is a finitely generated $\mathbb{F}_p$-algebra with good filtration for $G_{\mathbb{F}_p}$, for instance by taking for $\mathbf{k} \otimes_{\mathbb{F}_p} R$ the algebra C^D in Lemma 9.1. We may choose r so that $\mathrm{gr}\, A$ is a noetherian $\mathbf{k} \otimes_{\mathbb{F}_p} R^{(r)}$-module. Then by Friedlander and Suslin, whose theorem [6, Theorem 1.5, Remark 1.5.1] already

had the proper generality, we now know that $H^*(G_r, \operatorname{gr} A)^{(-r)}$ is a noetherian module over the graded algebra $\bigotimes_{i=1}^r S^*_{\mathbf{k}}((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes_{\mathbb{F}_p} R$. This graded algebra has good filtration. So our Theorem 1.1 tells there are only finitely many nonzero $H^i(G/G_r, H^*(G_r, \operatorname{gr} A))$ and they are all noetherian over $H^0(G/G_r, H^*(G_r, \operatorname{gr} A))$ by Corollary 1.2. In view of [7] the proof of Touzé in [18] goes through. □

Remark 10.4 Let G be a flat affine algebraic group scheme over a ring R. Suppose G_S satisfies (CFG) for some faithfully flat commutative R-algebra S. Then so does G. Therefore Theorem 10.1 has consequences for some twisted families.

Reductive group schemes over a noetherian base ring. Let $\mathbf{k}$ be a noetherian ring and let G be a reductive algebraic group scheme over $\operatorname{Spec}(\mathbf{k})$, in the sense of SGA3, as always. By [5, Exposé XXII, Corollaire 2.3] the group scheme G is locally split in the étale topology on $\operatorname{Spec}(\mathbf{k})$. Almost all the properties we try to establish are fpqc local on $\operatorname{Spec}(\mathbf{k})$, so in proofs we may and shall further assume G is split. Note that we have only defined the Grosshans filtration when the group is split.

In view of what we just did for the case that $\mathbf{k}$ contains a field, we may replace $\mathbb{Z}$ with any noetherian ring $\mathbf{k}$ in section 6 of [7]. Assume as always that the commutative algebra A is finitely generated over the noetherian ring $\mathbf{k}$, with rational action on A of G. When G is split, we provide A with the Grosshans filtration. Further, let M be a noetherian A-module with compatible G-action. An abelian group L has *bounded torsion* if there is an $n \geq 1$ with $nL_{\mathrm{tors}} = 0$. Summarizing section 6 of [7] we get

Theorem 10.5 (Provisional CFG) *We have*

- *Every $H^m(G, M)$ is a noetherian A^G-module.*
- *If $H^*(G, A)$ is a finitely generated $\mathbf{k}$-algebra, then $H^*(G, M)$ is a noetherian $H^*(G, A)$-module.*
- *If G is split, then $H^*(G, \operatorname{gr} A)$ is a finitely generated $\mathbf{k}$-algebra.*
- *$H^*(G, A)$ has bounded torsion if and only if it is a finitely generated $\mathbf{k}$-algebra.*

- *If $H^*(G,A)$ has bounded torsion, then the reduction $H^{\mathrm{even}}(G,A) \to H^{\mathrm{even}}(G,A/pA)$ is power-surjective for every prime number p.*

- *If $H^{\mathrm{even}}(G,A/pA)$ is a noetherian $H^{\mathrm{even}}(G,A)$-module for every prime number p, then $H^*(G,A)$ is a finitely generated $\mathbf{k}$-algebra.*

$\square$

Remark 10.6 If $\mathbf{k}$ contains $\mathbb{Q}$ or a finite ring then $H^*(G,A)$ obviously has bounded torsion. Also, if $H^i(G,A)$ vanishes for $i \gg 0$ then $H^*(G,A)$ has bounded torsion.

Remark 10.7 The $\mathbb{F}_p$ vector space $H^1(\mathbb{G}_a, \mathbb{F}_p)$ is infinite dimensional, so the hypothesis that G is a reductive group scheme can not be deleted.

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